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AN EXTENSION OF THE HAMILTON–JACOBI METHOD†

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A method of solving the canonical Hamilton equations, based on a search for invariant manifolds, which are uniquely projected onto position space, is proposed. These manifolds are specified by covector fields, which satisfy a system of first-order partial differential equations, similar in their properties to Lamb's equations in the dynamics of an ideal fluid. If the complete integral of Lamb's equations is known, then, with certain additional assumptions, one can integrate the initial Hamilton equations explicitly. This method reduces to the well-known Hamilton–Jacobi method for gradient fields. Some new conditions for Hamilton's equations to be accurately integrable are indicated. The general results are applied to the problem of the motion of a variable body.

1. INTRODUCTION

The Hamilton–Jacobi method reduces the problem of solving the canonical equations

$$x_i^{\cdot} = \partial H / \partial y_i, \quad y_i^{\cdot} = -\partial H / \partial x_i, \quad 1 \leqslant i \leqslant n; \quad H = H(x, y, t) \tag{1.1}$$

to an investigation of the first-order partial differential equation

$$\partial S / \partial t + H(x_1, \ldots, x_n, \partial S / \partial x_1, \ldots, \partial S / \partial x_n, t) = 0 \tag{1.2}$$

If $S(x, t)$ is a particular solution of Eq. (1.2), the relation

$$y = \partial S / \partial x \tag{1.3}$$

gives an n-dimensional invariant manifold Σ of system (1.1). Suppose $S(x, t, c)$, $c = (c_1, \ldots, c_n)$ is the complete integral of Eq. (1.2): for all c this function satisfies Eq. (1.2) and

$$\det \left\| \partial^2 S / \partial x_i \partial c_j \right\| \neq 0 \tag{1.4}$$

In this case the phase space of system (1.1) is stratified on invariant manifolds

$$\Sigma_c = \{x, y: y = \partial S / \partial x\}$$

where, by Jacobi's theorem, the following relations hold

$$\partial S / \partial c = -a, \quad a = (a_1, \ldots, a_n) \tag{1.5}$$

From (1.5) one can obtain the variable x as a function of t, and $2n$ arbitrary constants a and c. The variables y are then found from (1.3).

By condition (1.4), using the implicit function theorem, from n equations (1.3) one can find (at least locally) $c_1, \ldots, c_n$ as functions of x, y, t. These functions are independent integrals of Eqs (1.1), which are in the involution: $\{c_i, c_j\} = 0$. Conversely, if we know the n independent involutive integrals of Hamilton's equations (1.1), we can explicitly construct the complete integral of Eq. (1.2). Note also that Eqs (1.3) and (1.5) specify the canonical transformation $x, y \to c, a$ with the generating function S.

A more general approach to investigating Hamilton's equations (1.1) consists of replacing Eq. (1.2) by the system of first-order partial differential equations

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$$\partial u/\partial t + (\mathrm{rot}\, u)\, v = -\,\partial h/\partial x \tag{1.6}$$

where $u = (u_1, \ldots, u_n)$ are functions of x and t (the convector field in position space $\{x\}$), $\mathrm{rot}\, u = \partial u/\partial x - (\partial u/\partial x)^T$, $\partial u/\partial x = \| \partial u_i/\partial x_j \|$ is an $n \times n$ skew-symmetric matrix, and $v = (v_1, \ldots, v_n)^T$, $v_i = \partial H/\partial y_i \,|_{y=u}$ is a vector field in $\{x\}$ and $h(x, t) = H(x, u(x, t), t)$.

When $n = 3$ the rot of the field u corresponds uniquely to the matrix rot u, so that $(\mathrm{rot}\, u)\, v = (\mathrm{rot}\, u) \times v$. Equation (1.6) has the form of the well-known Lamb equation in the dynamics of an ideal fluid, and hence, in general, will also be called its *Lamb's equation*.

In order to understand the structure of Eqs (1.6) better, we will write them for systems with a "natural" Hamiltonian

$$H = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}(x,t)\, y_i y_j + V(x,t), \quad x = (x_1, \ldots, x_n)$$

Lamb's equations have the following explicit form

$$\frac{\partial u_i}{\partial t} + \sum_{j,k=1}^{n} g_{jk} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) u_k = -\frac{1}{2} \frac{\partial}{\partial x_i} \left(\sum_{j,k=1}^{n} g_{jk} u_j u_k \right) - \frac{\partial V}{\partial x_i}, \quad i = 1, \ldots, n$$

Equation (1.6) means that

$$\Sigma = \{ y = u(x, t) \}$$

is an n-dimensional invariant surface of system (1.1) [1]. If $u = \partial S/\partial x$, we have rot $u = 0$, and from (1.6) we obtain

$$\partial(\partial S/\partial t + h)/\partial x = 0$$

Consequently

$$\partial S/\partial t + H(x, \partial S/\partial x, t) = g(t)$$

After making the replacement

$$S \to S - \int g(t)\, dt$$

the function g is equal to zero. The derivation of (1.2) from (1.6) for invariant surfaces (1.3) is identical with the derivation of the Lagrange–Cauchy integral for potential flows of an ideal fluid. Hence, the invariant surfaces Σ will be called *potential (vortex) surfaces* if rot $u = 0$ (rot $u \neq 0$).

Equations (1.6) for Hamiltonian systems appeared for the first time, obviously, in the variational calculus as the conditions for extremal fields to match (see [2]). The extension of Lamb's equations to non-Hamiltonian systems can be found in [3]. The relation between Eq. (1.6) and the ideas from hydrodynamics can be found in [1, 4]. The use of Lamb's equations to integrate the equations of analytical mechanics is described in [3, 5], although it is families of potential invariant surfaces that are mainly considered there.

2. THE VORTEX METHOD OF INTEGRATING HAMILTON'S EQUATIONS

Suppose $u(x, t, c)$ is a family of solutions of Eqs (1.6), which depends on n parameters $c = (c_1, \ldots, c_n)$. We will call this family *the complete integral* of Lamb's equation (1.6), if

$$\det \| \partial u/\partial c \| \neq 0 \tag{2.1}$$

Theorem 1. Suppose we know the complete integral $u(x, t, c)$ of Eq. (1.6), where:

1. rank (rot u) $= 2k$.
2. k integrals $F_1(x, y, t), \ldots, F_k(x, y, t)$ of Hamilton's equations (1.1) exist such that $\{F_k, F_j\} = 0$ (for all $1 \leqslant i, j \leqslant n$) and for all values of c the field $u(x, t, c)$ satisfies each of the k equations

$$\partial u / \partial t + (\operatorname{rot} u) v_i = -\partial f_i / \partial x, \quad 1 \leq i \leq k \tag{2.2}$$

$$v_i = \partial F_i / \partial y|_{y=u}, \quad f_i(x,t,c) = F_i(x,u(x,t,c),t)$$

3. $f_1, \ldots, f_k$ are independent as functions of x.
Then the initial Hamilton equations (1.1) can be integrated in quadratures.

Note. Since the matrix rot u is skew-symmetric, its rank is an even number.

We will put out a number of corollaries of Theorem 1. We will first consider the case when u is a potential solution of Lamb's equation: $u = S_x'(x, t, c)$. Then rot $u = 0$ and consequently $k = 0$. In this case the condition for (2.1) to be non-degenerate becomes conditions (1.4), while Theorem 1 reduces to Jacobi's theorem on the complete integral of Eq. (1.2).

We will now consider the simplest of the vortex solutions of Eq. (1.6), when rank (rot u) = 2 and consequently $k = 1$.

If system (1.1) is autonomous (the Hamiltonian H is clearly independent of t), we can take the function H as the integral F from Section 2. The field u then obviously satisfies Eq. (2.2), since this equation is identical with the initial Lamb equation (1.6). Condition 3 of Theorem 1 reduces to the condition

$$d_x H(x, u(x, t, c)) \neq 0 \tag{2.3}$$

We have therefore established the following corollary.

Corollary. If we know the complete integral $u(x, t, c)$ of Eq. (1.6), defined by the Hamiltonian $H(x, y)$, where rank (rot u) = 2 and condition (2.4) is satisfied, Hamilton's equations (1.1) are integrable in quadratures.

This assertion is particularly effective when $n = 3$: the rank of the matrix rot u can be equal to either zero or two.

Finally, we will consider another limiting case, when the matrix of the rot u has the maximum possible rank, equal to n. Then $n = 2k$, and for the complete integration of Hamilton's equations we must know $n/2$ involutive integrals which satisfy conditions 2 and 3 of Theorem 1. We can give these conditions a clear interpretation in the autonomous case when the functions $H = F_1, \ldots, F_{n/2}$ and the field u are explicitly independent of t. Then Lamb's equation (1.6)

$$(\operatorname{rot} u)\, x^{\cdot} = -\partial h / \partial x \tag{2.4}$$

will be a Hamiltonian system in $2k$-dimensional phase space $\{x\}$ with the simplectic structure

$$\omega = d(udx) = \sum (\partial u_i / \partial x_j - \partial u_j / \partial x_i)\, dx_i \wedge dx_j$$

and Hamiltonian h. By conditions 2 and 3 the functions $h = f_1, \ldots, f_k$ are independent integrals of Eq. (2.5). Since $\{F_i, F_j\} = 0$, the functions $f_1, \ldots, f_k$ are also involutive with respect to the simplectic structure ω. By Liouville's theorem, Eqs (2.5) are integrable in quadratures. The momenta y are found from the relations $y = u(x, t, c)$.

Theorem 1 is proved in Sections 3 and 4. Equations (1.1) are first reduced to an autonomous system in $(2n + 2)$-dimensional phase space, and the autonomous version of Theorem 1 is then proved.

3. REDUCTION TO THE AUTONOMOUS CASE

It is well known that the non-autonomous Hamiltonian system (1.1) with n degrees of freedom can be represented in the form of an autonomous system with $n + 1$ degrees of freedom, by adding the conjugate variables $x_{n+1} = t, y_{n+1}$ to the canonical variables x, y and introducing the new Hamiltonian

$$H^* = y_{n+1} + H(x, y, x_{n+1}) \tag{3.1}$$

Equations (1.1) are equivalent to the system

$$x^{\cdot} = \partial H^* / \partial y, \quad y^{\cdot} = -\partial H^* / \partial x$$

The other two equations have the form

$$x_{n+1}^{\cdot} = \partial H^* / \partial y_{n+1} = 1, \quad y_{n+1}^{\cdot} = -\partial H^* / \partial x_{n+1} = -\partial H / \partial t$$

The first of these is a trivial identity, while the second (taking the integral $H^* = \text{const}$ into account) represents a theorem on the change in the energy H for system (1.1). Suppose system (1.1) has an n-dimensional invariant manifold $y = u(x, t)$. Then the relations

$$y = u(x, x_{n+1}), \quad y_{n+1} = u_{n+1}(x, x_{n+1}) = -H(x, u, x_{n+1})$$

specify an $(n + 1)$-dimensional invariant manifold for a system with Hamiltonian (3.1).

We will put

$$x^* = (x, x_{n+1}), \quad u^* = (u, u_{n+1}), \quad y^* = (y, y_{n+1})$$

$$\operatorname{rot} u^* = \partial u^* / \partial x^* - (\partial u^* / \partial x^*)^T$$

$$v^* = (x^{\cdot}, x_{n+1}^{\cdot})_{y=u}^T = (v, 1)^T, \quad h^* = H^*|_{y^*=u^*} = 0$$

It can be shown that the autonomous Lamb equations for a Hamiltonian system with Hamilton function H^*

$$(\operatorname{rot} u^*) v^* = -\partial h^* / \partial x^* = 0 \tag{3.2}$$

are equivalent to Eqs (1.6).

The complete integral $u(x, t, c)$ of Lamb's equations (1.6) can be extended to the complete integral of Eqs (3.2) by assuming

$$y_{n+1} = -H(x, u(x, x_{n+1}, c), x_{n+1}) + c_{n+1}, \quad c_{n+1} = \text{const}$$

In fact

$$\det \| \partial u^* / \partial c^* \| = \det \| \partial u / \partial c \| \neq 0, \quad c^* = (c_1, \ldots, c_{n+1})$$

Thus, we can confine ourselves to considering autonomous systems (1.1) with Hamiltonian $H(x, y)$ and steady convector fields $u(x)$. These objects are related by Eqs (2.5), in which $h(x) = H(x, u(x))$.

Theorem 2. Suppose we know an n-parametric solution $u(x, c)$ of Eqs (2.5), which satisfy condition (2.1) and the following conditions:

1. rank $(\operatorname{rot} u) = 2k$
2. there are k involute integrals $F_1(x, y), \ldots, F_k(x, y)$ of autonomous system (1.1) such that for all values of c

$$(\operatorname{rot} u) v_i = -\partial f_i / \partial x, \quad 1 \leqslant i \leqslant k$$

$$v_i = \partial F_i / \partial y |_{y=u}, \quad f_i(x, c) = F_i(x, u(x, c))$$

3. the functions $f_1, \ldots, f_k$ are independent.

Then system (1.1) can be integrated using quadratures.

This assertion is a consequence of Theorem 1 in the special case when the functions $H, F_1, \ldots, F_k$ and the field u are explicitly independent of t. On the other hand, Theorem 1 is derived from Theorem 2 using the extension of phase space described above. The functions $F_1(x, y, x_{n+1}), \ldots, F_k(x, y, x_{n+1})$ play the role of k integrals of condition 2 of Theorem 2.

A unique interesting point which is required in the proof is equality of the ranks of the skew-symmetric matrices $A = \operatorname{rot} u$ and $B = \operatorname{rot} u^*$. We recall that the rank of a matrix is equal to the codimension of its zero-space, consisting of the eigenvectors with zero eigenvalue. These vectors are also called *vortex* vectors.

Suppose

$$\lambda_1 = (\lambda_{11},\dots,\lambda_{1n})^T,\dots,\ \lambda_m = (\lambda_{m1},\dots,\lambda_{mn})^T \tag{3.3}$$

are linearly independent vortex vectors of the matrix A and $m = n - \operatorname{rank} A$. It is clear that the vectors

$$\lambda_i = (\lambda_{i1},\dots,\lambda_{in},0)^T,\quad 1 \leqslant i \leqslant m \tag{3.4}$$

will be linearly independent vortex vectors of the matrix B. It remains to note that, according to (3.2), B has one other vortex vector $v^* = (v_1,\dots,v_n,1)^T \neq 0$, linearly independent with vectors (3.4).

The vortex vectors (3.3) of the matrix rot u depend on the point x, and their linear combinations generate an m-dimensional distribution $\Pi(x)$. It turns out [4] that this distribution is integrable: the n-dimensional space of positions $\{x\}$ is stratified into m-dimensional surfaces, the tangential planes of which coincide with $\Pi(x)$ at the point x. Consequently, the local coordinates $x_1,\dots,x_n$ can be chosen so that the integral surfaces of the distribution Π (*the vortex manifolds*) are specified by the equations

$$x_1 = \alpha_1,\dots,\ x_{2k} = \alpha_{2k};\quad \alpha = \text{const},\quad 2k = n - m$$

We can take as the vortex vectors (3.3)

$$\lambda_1 = (0,\dots,0,1,0,\dots,0)^T,\dots,\ \lambda_m = (0,\dots,0,\dots,1)^T$$

Since $(\text{rot}\,u)\,\lambda_i = 0$ $(1 \leqslant i \leqslant m)$, we have $\partial u_i/\partial x_j = \partial u_j/\partial x_i$ for all $i = 1,\dots,n$ and $j = n-m+1,\dots,n$. In particular

$$\partial u_p / \partial x_j = \partial u_j / \partial x_p,\quad \partial u_q / \partial x_j = \partial u_j / \partial x_q,\quad 1 \leqslant p,\ q \leqslant 2k$$

Consequently

$$(\partial / \partial x_j)(\partial u_p / \partial x_q - \partial u_q / \partial x_p) = 0$$

Hence, the matrix rot u has the following form: its last $n - 2k$ columns and $n - 2k$ rows are zero, while the remaining elements form a skew-symmetric $2k \times 2k$-matrix

$$(\text{rot}\,u)_* = \left\| \partial u_p / \partial x_q - \partial u_q / \partial x_p \right\|;\quad p,q \leqslant 2k$$

the elements of which are independent of $x_{2k+1},\dots,x_n$.

It has been proved [4] that the function h is constant on vortex manifolds. So in these variables it depends only on $x_1,\dots,x_{2k}$. Hence, Eq. (2.4) reduces to the equation

$$(\text{rot}\,u)_* \dot{x}_* = -\partial h / \partial x_*,\qquad x_* = (x_1,\dots,x_{2k}) \tag{3.5}$$

with non-degenerate matrix (rot u). Since the equations from condition 2 of Theorem 2 are identical in form with (2.4), the functions $f_1,\dots,f_k$ are also constant on vortex manifolds and are independent of $x_{2k+1},\dots,x_n$. These functions comprise a complete involute set of independent integrals of Hamilton's equations (3.5), and hence Liouville's theorem of complete integrability applies.

It was proved in [1] that the phase flux of system $\dot{x} = v(x)$ converts vortex manifolds. Consequently, the components $v_{2k+1},\dots,v_n$ of the field v are independent of the variables $x_{2k+1},\dots,x_n$, and these variables can be found as functions of t by simple quadratures.

Elimination of the variables $x_{2k+1},\dots,x_n$ indicates factorization on the space of positions $\{x\}$ with respect to the manifolds: the points x lying on one vortex manifold are identified. From this point of view, system (3.5) is a *factor-system* (2.5) for this equivalence ratio. Hence, the problem of integrating system (1.1) rests on the problem of constructing a family of vortex manifolds.

4. PROOF OF THEOREM 2

Consider the family of surfaces of compatible levels of the functions f_i

$$M_\beta^{n-k} = \{x\colon f_1(x) = \beta_1,\dots,f_k(x) = \beta_k\},\quad \beta = \text{const}$$

Suppose $V_1, \ldots, V_k$ are Hamiltonians of vector fields in $2n$-dimensional phase space of the variables x and y, generated by the Hamiltonians $F_1, \ldots, F_k$. Since $\{F_i, F_j\} = 0$, these fields commute pairwise: $[V_i, V_j] = 0$. By condition 2, the fields V_i touch the n-dimensional surfaces $\Sigma_c = \{x, y\colon y = u\}$ and hence the projections $v_1, \ldots, v_k$ of these fields are correctly defined in the configuration space $\{x\}$. Since the fields V_i commute pairwise, we have $[v_i, v_j] = 0$.

Since $\{F_i, F_j\} = 0$, each function F_i is an integral of the vector field V_j: $V_j(F_i) = 0$. Consequently, $v_j(f_i) = 0$ for all $i, j = 1, \ldots, k$. This indicates that the fields $v_1, \ldots, v_k$ touch each surface M^{n-k}.

On the other hand, there are independent vortex vector fields $w_1, \ldots, w_{n-2k}$ which also touch M^{n-k}. In fact, by (3.4), $w_j(f_i) = 0$. Further, the vectors

$$v_1, \ldots, v_k, \quad w_1, \ldots, w_{n-2k} \tag{4.1}$$

are linearly independent. Otherwise

$$\Sigma \lambda_i v_i + \Sigma \mu_j w_j = 0 \tag{4.2}$$

with certain λ_i and μ_i, where $\Sigma \mid \lambda_i \mid \neq 0$. Multiplying (4.2) on the left by rot u and using condition 2, we obtain

$$\Sigma \lambda_i (\operatorname{rot} u) v_i = -\Sigma \lambda_i \partial f_i / \partial x = 0$$

However, by condition 3 of the theorem, the functions $f_1, \ldots, f_k$ are independent. Consequently, all $\lambda_i = 0$. We have obtained a contradiction. Note that the number of independent tangential fields (4.1) is identical with the dimension of the manifold M^{n-k}.

We will now obtain the $(n-2k)$-dimensional vortex manifolds rot u, or more correctly, the intersection of these manifolds with the $(n-k)$-dimensional surfaces M^{n-k}. They are k-dimensional and hence are specified by M^{n-k} equations

$$\varphi_1(x) = \gamma_1, \ldots, \quad \varphi_k(x) = \gamma_k, \quad x \in M^{n-k}$$

By the definition of vortex manifolds, the functions φ_i satisfy the equations

$$w_1(\varphi_i) = \ldots = w_{n-2k}(\varphi_i) = 0, \quad 1 \leqslant i \leqslant k \tag{4.3}$$

We will seek these from the additional conditions

$$v_j(\varphi_i) = \delta_{ji}, \quad 1 \leqslant j \leqslant k \tag{4.4}$$

where δ_{ji} is the Kronecker delta.

It is first necessary to show that the systems of first-order partial differential equations (4.3) and (4.4) have solutions. In fact, $[v_i, v_j] = 0$, and it can be shown that the commutators can be expressed linearly in terms of the vortex vectors w [4]. Under these conditions the solvability of system (4.3)–(4.4) follows from the well-known results of the theory of solvable algebras of vector fields (see, for example, [6]).

Since the vector fields (4.1) are independent, from (4.3) and (4.4) we can uniquely obtain (using only algebraic operations) the partial derivatives of the function φ_i with respect to local coordinates on M^{n-k}. It remains to use well-known quadratures, which recover from the function from its derivatives. The theorem is proved.

5. THE RELATION TO THE THEORY OF NON-COMMUTATIVE INTEGRATION

Suppose $u(x, t, c)$ is the complete integral of Lamb's equations (1.6). Since condition (2.1) is satisfied, using the implicit function theorem the system of equations

$$y_i = u_i(x, t, c_1, \ldots, c_n), \quad 1 \leqslant i \leqslant n \tag{5.1}$$

can be solved (at least locally) for the parameters c

$$c_1 = F_{k+1}(x,y,t),\ldots,\quad c_n = F_{n+k}(x,y,t) \tag{5.2}$$

In view of the invariance of the surfaces (5.1) these functions are integrals of Hamilton's equations (1.1) [1]. By condition 3 of Theorem 1, the functions $F_1, \ldots, F_k, \ldots, F_{n+k}$ are independent. By condition 2, the first k functions commute with all the remaining ones. Consequently, the rank r of the matrix of the Poisson brackets

$$\|\{F_i, F_j\}\|, \quad 1 \leqslant i,j \leqslant n+k \tag{5.3}$$

is identical with the rank of the matrix of the Poisson brackets of functions (5.2). It was shown in [7] that this number is identical with the rank of the rot u, i.e. it is equal to $2k$.

Thus, the number $m = n + k$ of known integrals of Hamilton's equations (1.1) is linked with the rank r of matrix (5.3) by the relation

$$2m = 2n + r \tag{5.4}$$

which is known as the condition of *non-commutative integrability* of system (1.1) [8]. When $r = 0$, condition (5.4) reduces to the condition of complete integrability of Hamilton's equations with n degrees of freedom.

Note that in the theory of non-commutative integration autonomous systems and closed sets of integrals are usually considered: their Poisson brackets $\{F_i, F_j\}$ are functions of F_s. With these assumptions it was proved in [9] that Hamilton's equations, which satisfy condition (5.4), are integrable in quadratures.

Theorem 3. We will assume that Hamilton's equations (1.1) have $n + k$ independent integrals

$$F_1(x,y,t),\ldots,\quad F_{n+k}(x,y,t) \tag{5.5}$$

where the first $n - k$ of these are in the involution with all the functions (5.5). Then Eqs (1.1) can be integrated in quadratures.

When $k = 0$ we obtain Liouville's theorem on the complete integrability of Hamiltonian systems. We will consider the autonomous case and assume that the surfaces of the common levels of integrals (5.5) are compact. It was proved in [10] that the coupled components of these surfaces will be $(n - k)$-dimensional tori with conditionally periodic motions, and in the neighbourhood of these tori one can introduce generalized action-angle variables. Note that in the assumptions of Theorem 3 the rank of matrix (5.3) is equal to $2k$. Consequently, condition (5.4) is satisfied, but it is not assumed here that the set of integrals (5.5) is closed.

To prove Theorem 3 we will consider the algebraic system of equations

$$F_{k+1}(x,y,t) = c_1,\ldots,\quad F_{n+k}(x,y,t) = c_n \tag{5.6}$$

and we will additionally assume that

$$\frac{\partial(F_{k+1},\ldots,F_{n+k})}{\partial(y_1,\ldots,y_n)} \neq 0 \tag{5.7}$$

Then we can solve system (5.6) for the canonical momenta: $y = u(x, t, c)$. Since for all values of c Eqs (5.6) specify the invariant surface of system (1.1), the field u satisfies Lamb's equation (1.6). Since the functions (5.6) are independent and condition (5.7) is satisfied, the family of solutions $u(x, t, c)$ satisfies condition (2.1). Consequently, this is a complete integral of Lamb's equation. The rank of matrix (5.3) is identical [7] with the rank of the matrix rot u. Hence, condition 1 of Theorem 1 is satisfied. Since $n - k \geqslant k$, the first k functions of the set (5.5) commute with all the functions (5.6). Hence, condition 2 of Theorem 1 is satisfied. Finally, condition 3 follows from the assumption that the set of functions (5.5) is independent. Hence, the integrability of Hamilton's equations (1.1) follows from Theorem 1.

If assumption (5.7) is not satisfied, the variables $y_1, \ldots, y_n$ must be replaced by other canonical coordinates. All the discussion remains the same apart from Lamb's equations that will have a somewhat different form.

6. APPLICATION TO THE DYNAMICS OF A VARIABLE BODY

As an example we will consider Liouville's problem [11] of the inertial rotation of a variable body: due to internal forces its particles are displaced with respect to one another. Euler's dynamic equations have the form

$$K^{\cdot} = [K, \omega], \quad K = I\omega + \lambda \tag{6.1}$$

where ω is the angular velocity of the body about its principal axes of inertia, $I = \mathrm{diag}(I_1, I_2, I_3)$ is the inertia matrix, and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the gyroscopic moment. We will assume that I_i and λ_i are known functions of time. In this case Eqs (6.1) will be closed.

Note. Other formulations of the problem are possible in the dynamics of a variable body. For example, Zeiliger and Chetayev [12] considered a similarly variable body and for the closure of system (6.1) they added an equation for the rate of "radiant" expansion.

Combining Poisson's equations for the unit fixed vectors α, β, γ

$$\alpha^{\cdot} = [\alpha, \omega], \quad \beta^{\cdot} = [\beta, \omega], \quad \gamma^{\cdot} = [\gamma, \omega] \tag{6.2}$$

with (6.1) we obtain a complete system for determining the orientation of the principal axes of inertia of the body. Equations (6.1) and (6.2) admit of three integrals

$$(K, \alpha) = c_1, \quad (K, \beta) = c_2, \quad (K, \gamma) = c_3 \tag{6.3}$$

A consequence of these is the integral of the momentum $(K, K) = k^2$ of Euler's equations (6.1).

We will seek three-dimensional invariant surfaces, which are uniquely projected onto configuration space—the SO(3) group. This means that the moment of momentum K must be sought in the form of a function of α, β, γ and the variable t. Then from (6.1) and (6.2) we obtain a vector partial differential equation

$$\frac{\partial K}{\partial t} + \frac{\partial K}{\partial \alpha}[\alpha, \omega] + \frac{\partial K}{\partial \beta}[\beta, \omega] + \frac{\partial K}{\partial \gamma}[\gamma, \omega] = [K, \omega] \tag{6.4}$$

Here we must substitute $I^{-1}(K - \lambda)$ instead of ω. This equation is, of course, Lamb's equation (1.6), except that it is not represented in canonical variables. Changing from (1.6) to (6.4) is completely analogous to changing from Hamilton's equations to the Poincaré–Chetayev equations in Lie algebras.

By (6.3), one of the complete solutions of Lamb's equation (6.4) will be a function of $K = c_1\alpha + c_2\beta + c_3\gamma$. It can be shown that the condition for (2.1) to be non-degenerate is satisfied and the rank of the matrix of the rot is equal to two, if the inertia operator I is not spherical.

We will assume that Eqs (6.1) have an integral $F(K, t)$ independent of the integral of the momentum K^2. Then Eqs (6.1)–(6.2) can be integrated in quadratures. This fact can be derived from Theorem 1.

In fact, here $k = 1$ and the Poincaré–Chetayev equation

$$K^{\cdot} = [K, \partial F/\partial K] \tag{6.5}$$

corresponds to the function F, to which we must add Poisson's equation (6.2). Clearly each surface (6.3) will be invariant for Eqs (6.5) and (6.2). Hence, condition 2 of Theorem 1 is satisfied. Condition 3 follows from the assumption that the functions F and K^2 are independent.

Note. The result that Eqs (6.1) and (6.2) are integrable in quadratures also follows from Theorem 3: the required set of integrals is comprised of the functions F and K^2, (K, α), (K, β). The integrability of the non-autonomous system (6.1) with the additional integral F also follows from the Euler–Jacobi theorem, since the divergence of the right-hand side of (6.1) is equal to zero.

It can be shown [4] that the vector fields in group $SO(3)$, which generate rotation of the axes of the inertia with an angular velocity that is constant in a fixed space, will be vortex fields. In particular, all the vortex lines are closed and the stratification of the $SO(3)$ group by the vortex lines are closed and the stratification of the SO(3) group by the vortex lines coincides with the well-known Hopf stratification. The corresponding factor space will be a Poisson sphere.

To fix our ideas we will consider the case when the vector of the moment of momentum K is directed along γ: $K = k\gamma, k = |K|$. Assuming the vector γ to be vertical, we will introduce the Euler angles, which specify the orientation of the principal axes of inertia of the variable body. We must put $c_1 = c_2 = 0$, $c_3 = k$ in (6.3). Using Euler's kinematic formulae, for these quantities we can write the equations of motion in the $SO(3)$ group

$$
\begin{aligned}
\vartheta^{\cdot} &= k(I_1^{-1} - I_2^{-1}) \sin\vartheta \sin\varphi \cos\varphi - \lambda_1 I_1^{-1} \cos\varphi + \lambda_2 I_2^{-1} \sin\varphi \\
\vartheta^{\cdot} &= k\cos\vartheta\left(\frac{1}{I_3} - \frac{\sin^2\varphi}{I_1} - \frac{\cos^2\varphi}{I_2}\right) + \frac{\lambda_1 \sin\varphi\cos\vartheta}{I_1 \sin\vartheta} + \frac{\lambda_2 \cos\varphi\cos\vartheta}{I_2 \sin\vartheta} - \frac{\lambda_3}{I_3} \\
\psi^{\cdot} &= k\left(\frac{\sin^2\varphi}{I_1} + \frac{\cos^2\varphi}{I_2}\right) - \frac{\lambda_1 \sin\varphi}{I_1 \sin\vartheta} - \frac{\lambda_2 \cos\varphi}{I_2 \sin\vartheta}
\end{aligned}
\tag{6.6}
$$

They allow of the integral invariant

$$\mathrm{mes}(D) = \iiint_D \sin\vartheta \, d\vartheta \, d\varphi \, d\psi$$

which is identical with the double-sided invariant Haar measure in the SO(3) group [13]. In these variables the vortex fields have the form $\vartheta' = 0$, $\varphi' = 0$, $\psi' = \mu$, while the vortex lines are specified by the equations ϑ, φ = const. Since the third equation of (6.6) does not contain the angle ψ explicitly, factorization with respect to the vortex lines leads to the first two equations of system (6.6). These are a closed Hamiltonian system on the Poisson sphere, and the standard 2-form of the area plays the part of the simplectic structure.

7. THE CLEBSCH POTENTIALS

As shown in Section 3, the fundamental difficulty in the explicit integration of Hamilton's equations for a known complete integral of Lamb's equations is finding the vortex manifolds. This problem can be simplified considerably if the covector field u is represented in the form of a sum

$$\partial S / \partial x + A_1 \partial B_1 / \partial x + \ldots + A_k \partial B_k / \partial x \tag{7.1}$$

where $S, A_1, B_1, \ldots$ are certain functions of x and t. In view of the formulae A_s and B_s have the same meaning. In hydrodynamics the functions $S, A_1, B_1, \ldots$ are usually called Clebsch potentials [14, Section 167].

If the potentials $A_1, B_1, \ldots, B_k$ are independent as functions of x, we have rank (rot u) = $2k$. Since

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \Sigma\left(\frac{\partial A_s}{\partial x_j}\frac{\partial B_s}{\partial x_i} - \frac{\partial A_s}{\partial x_i}\frac{\partial B_s}{\partial x_j}\right)$$

the vortex vectors coincide with the tangential vectors to the $(n-2k)$-dimensional surfaces

$$\{x:\ A_1(x,t) = a_1,\ \ B_1(x,t) = b_1, \ldots, B_k(x,t) = b_k\},\quad a, b = \text{const}$$

Consequently, these surfaces are the required vortex manifolds.

By Darboux's theorem [15], the Clebsch potentials always exist. Moreover, the functions $A_1, B_1, \ldots, B_k$ can be taken as new coordinates; we will denote them by $x_1, \ldots, x_{2k}$. Expanding this point transformation to a linear canonical transformation, we can write formulae (7.1) in explicit form

$$u_1 = \frac{\partial S}{\partial x_1},\quad u_2 = \frac{\partial S}{\partial x_2} + x_1, \ldots,\quad u_{2k+1} = \frac{\partial S}{\partial x_{2k+1}}, \ldots,\quad u_n = \frac{\partial S}{\partial x_n}$$

and Lamb's equations (1.6)

$$x_1^{\cdot} = -\frac{\partial}{\partial x_2}\left(\frac{\partial S}{\partial t} + h\right), \quad x_2^{\cdot} = \frac{\partial}{\partial x_1}\left(\frac{\partial S}{\partial t} + h\right) \tag{7.2}$$

....

$$x_{2k-1}^{\cdot} = -\frac{\partial}{\partial x_{2k}}\left(\frac{\partial S}{\partial t} + h\right), \quad x_{2k}^{\cdot} = \frac{\partial}{\partial x_{2k-1}}\left(\frac{\partial S}{\partial t} + h\right)$$

$$\frac{\partial}{\partial x_{2k+1}}\left(\frac{\partial S}{\partial t} + h\right) = \ldots = \frac{\partial}{\partial x_n}\left(\frac{\partial S}{\partial t} + h\right) = 0 \tag{7.3}$$

It follows from (7.3) that $\partial S/\partial t + h$ is a function of the coordinates $x_1, \ldots, x_{2k}$ and the time t only. This relation extends the Hamilton–Jacobi equation and reduces to it when $k = 0$. Then (7.2) will be a closed canonical system of differential equations for the Clebsch potentials with Hamiltonian $\partial S/\partial t + h$. These observations extend the well-known results obtained by Clebsch and Stewart [14] to vortex flows of an ideal fluid (when $n = 3$).

We will now assume that the conditions of Theorem 1 are satisfied. It can be shown that (by (2.3)) the functions $f_1, \ldots, f_k$ are independent of $x_{2k+1}, \ldots, x_n$ and are in involution. Hence, these functions form a complete set of independent involutive integrals of Hamilton's equations (7.2). The explicit solution of Eqs (7.2) can be obtained using the construction of the complete integral of the Hamilton–Jacobi equation for the Hamiltonian $\partial S/\partial t + h$. After this, the remaining variables $x_{2k+1}, \ldots, x_n$ are found by simple quadratures (see Section 3).

As an example we will again consider the rotation of the axes of inertia of a variable body. Suppose $p_\vartheta, p_\varphi, p_\psi$ are canonical variables, conjugate to the Euler angles ϑ, φ, ψ. Taking the axis of the constant moment of momentum of the body to be vertical, we write the equation of the three-dimensional invariant surface in these variables, the surface being uniquely projected onto $SO(3)$.

$$p_\psi = k, \quad p_\vartheta = 0, \quad p_\varphi = k\cos\vartheta$$

We can take as the Clebsch potentials

$$S = k\psi, \quad A = k\cos\vartheta, \quad B = \varphi$$

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